



The Open
University

MST124

Essential mathematics 1

Exercise Booklet 5

1 Distance

Exercise 1

Find the distance between the two points in each of the following pairs.

- (a) $(-1, 1)$ and $(4, 3)$
 - (b) $(0, 6)$ and $(2, -2)$
 - (c) $(-7, -4)$ and $(-2, -1)$
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Exercise 2

Find the midpoint of the line segment joining the points $(-7, -4)$ and $(-2, -1)$.

Exercise 3

Find the equation of the perpendicular bisector of the line segment joining the points $(0, 6)$ and $(6, -2)$.

2 Circles

Exercise 4

Write down the equations of the circles with the following centres and radii.

- (a) centre $(3, 2)$, radius 5
 - (b) centre $(-\frac{1}{2}, 0)$, radius $\frac{2}{3}$
 - (c) centre $(-1, -1)$, radius $\sqrt{7}$
-

Exercise 5

For each of the following equations, write down the centre and radius of the circle that it represents.

- (a) $(x - 6)^2 + (y + 3)^2 = 121$
 - (b) $3x^2 + 3y^2 = 9$
 - (c) $\frac{(x + k)^2}{2} + \frac{(y + k)^2}{2} = h$.
-

Exercise 6

Show that each of the following equations represents a circle, and find the centre and radius in each case.

- (a) $x^2 + y^2 + 10y = 0$
 - (b) $x^2 + 6x + y^2 - 5y = -13$
 - (c) $2x^2 - 4x + 2y^2 + 4y - 1 = 0$
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Exercise 7

Which of the following equations represent a circle? For each equation that represents a circle, find the centre and radius of the circle.

- (a) $(x - 3)^2 + y^2 = -5$
 - (b) $x^2 - y^2 = 25$
 - (c) $(x + 5)^2 + (2y - 6)^2 = 100$
 - (d) $x^2 + y^2 - 6x = -5$
 - (e) $10x^2 + 20x + 10y^2 - 20y = 0$
 - (f) $x^2 + 14x + y^2 + 14y + 98 = 0$
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Exercise 8

Find the following points on the circle $x^2 + 4x + y^2 - 8y = 5$, where they exist.

- (a) The points with x -coordinate 2.
 - (b) The points with x -coordinate 3.
 - (c) The points with y -coordinate -2 .
 - (d) The points with y -coordinate 0.
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Exercise 9

For each of the following sets of points, find the equation of the circle that passes through them.

- (a) $A(-3, 0)$, $B(5, 4)$ and $C(2, 10)$
 - (b) $A(-1, 3)$, $B(1, -7)$ and $C(5, 13)$
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3 Points of intersection

Exercise 10

For each of the following lines, either find the point or points at which the line intersects the circle $(x+1)^2 + (y-4)^2 = 9$, or show that the line does not intersect the circle.

- (a) $x = 2$ (b) $y = x$ (c) $y = x + 2$

Exercise 11

Find the points (if any) at which the following line and parabola intersect:

$$y = x - 1$$

$$y = x^2 + 2x - 3.$$

Exercise 12

Find the points (if any) at which the following parabolas intersect:

$$y = 2x^2 + x - 4$$

$$y = 2x^2 - x.$$

Exercise 13

Find the points (if any) at which the following two circles intersect:

$$(x+2)^2 + (y+1)^2 = 26$$

$$(x-3)^2 + (y-4)^2 = 16.$$

4 Working in three dimensions

Exercise 14

Find the distance between the two points in each of the following pairs.

- (a) $(0, 7, 2)$ and $(3, 3, 14)$

- (b) $(8, -4, -2)$ and $(4, -1, 3)$

Exercise 15

Write down the standard form of the equation of the sphere with centre at $(0, 10, -4)$ and radius 9.

Exercise 16

Write down the centre and radius of the sphere with equation

$$(x+4)^2 + (y-7)^2 + z^2 = 121.$$

Exercise 17

Which of the following equations represent spheres? For each equation that represents a sphere, find the centre and radius of the sphere.

(a) $x^2 + y^2 + z^2 - 2x + 14y = 50$

(b) $2x^2 - y^2 + 3z^2 - 2x + 8y + 6z = 73$

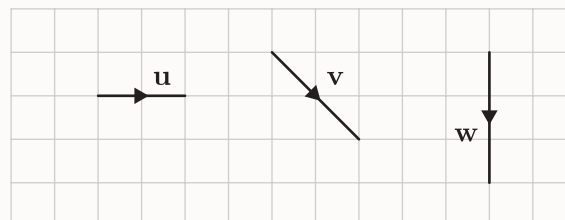
(c) $x^2 + y^2 + z^2 - 4x + 6y + 10z + 100 = 0$

(d) $5x^2 + 5y^2 + 5z^2 + 10x - 20y + 30z = 0$

5 Vectors

Exercise 18

The diagram below shows three vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ drawn on a grid.



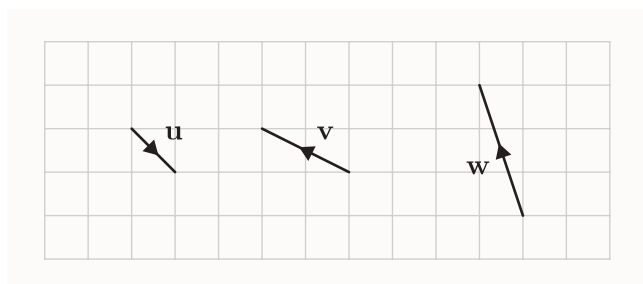
Draw diagrams showing the following vectors.

- (a) $\mathbf{u} + \mathbf{w}$ (b) $2\mathbf{u}$ (c) $-\mathbf{v}$

- (d) $\mathbf{w} - \mathbf{v}$

Exercise 19

The diagram below shows three vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ drawn on a grid.



Draw diagrams showing the following vectors.

- (a) $\mathbf{v} + \mathbf{w}$ (b) $\mathbf{w} - \mathbf{u} + \mathbf{v}$ (c) $2\mathbf{v} + 3\mathbf{u}$
 (d) $-2\mathbf{w} - \mathbf{v}$

Exercise 20

Simplify the vector expression

$$7(\mathbf{a} + \mathbf{b}) - (\mathbf{a} - \mathbf{c}) + 6(\mathbf{b} - \mathbf{a} + 2\mathbf{c}).$$

Exercise 21

Given that the vectors $\mathbf{x}$, $\mathbf{a}$ and $\mathbf{b}$ satisfy the vector equation

$$3(\mathbf{a} + 2\mathbf{x}) = 4(\mathbf{b} - \mathbf{a}) + \mathbf{a},$$

express $\mathbf{x}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

Exercise 22

An orienteer runs 240 m on a bearing of 320° . She then turns and runs 70 m on a bearing of 50° . Find the magnitude and bearing (to the nearest degree) of her resultant displacement.

Exercise 23

A ball in a computer game travels 15 cm on a bearing of 110° , then hits an object and continues for a further 12 cm on a bearing of 240° .

Find the resultant displacement of the ball, giving the magnitude in cm correct to one decimal place and the bearing to the nearest degree.

Exercise 24

The displacement from Glasgow to Edinburgh is 69 km with a bearing of 81° . The displacement from Edinburgh to Stirling is 50 km with a bearing of 292° . Find the magnitude (to the nearest km) and direction (as a bearing to the nearest degree) of the displacement from Glasgow to Stirling.

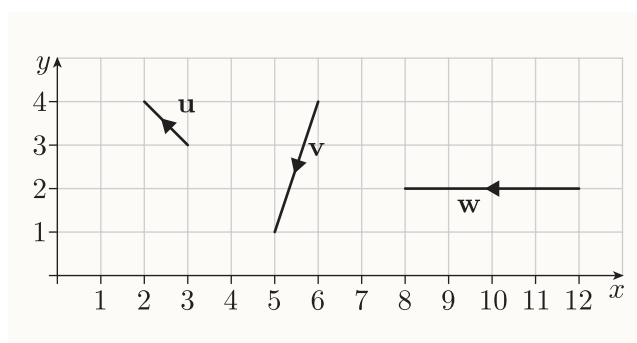
Exercise 25

A bird has a speed in still air of 15 m s^{-1} . It is pointed in the direction of the bearing 140° but it flies in a wind of speed 8 m s^{-1} blowing from the south. Find the resultant velocity of the bird in terms of its speed (in m s^{-1} to one decimal place) and bearing (to the nearest degree).

6 Component form of a vector

Exercise 26

Express each of the vectors below in component form.



Exercise 27

Find each of the following vectors, where $\mathbf{a} = \mathbf{i} + 2\mathbf{j}$, $\mathbf{b} = -3\mathbf{i} + \mathbf{j}$ and $\mathbf{c} = -2\mathbf{i} - 5\mathbf{j}$.

- (a) $\mathbf{a} + \mathbf{b}$ (b) $\mathbf{a} - \mathbf{b} - \mathbf{c}$ (c) $2\mathbf{c}$
 (d) $\frac{1}{3}\mathbf{b}$ (e) $\mathbf{c} - 3\mathbf{b}$ (f) $2\mathbf{a} + 5\mathbf{b} - 7\mathbf{c}$

Exercise 28

Write each of the following vector expressions as a single vector.

$$(a) -3 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (b) 4 \begin{pmatrix} -2 \\ 5 \end{pmatrix} - 2 \begin{pmatrix} 3 \\ -7 \end{pmatrix}$$

$$(c) 2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - 3 \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Exercise 29

Let A be the point $(-2, -1)$ and B be the point $(5, 0)$.

- (a) Find, in component form, the vector from A to B .
- (b) Find, in component form, the vector from B to A .

Exercise 30

For each of the following vectors, find its magnitude (as an exact value) and the angle (to the nearest degree) that it makes with the positive x -direction.

$$(a) \mathbf{a} = \mathbf{i} + \mathbf{j} \quad (b) \mathbf{b} = 7\mathbf{i} - 8\mathbf{j}$$

$$(c) \mathbf{c} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

Exercise 31

Find the magnitude and bearing of each of the following velocity vectors, which are given in ms^{-1} , with $\mathbf{i}$ pointing east and $\mathbf{j}$ pointing north. Give the magnitudes in ms^{-1} to one decimal place and the bearings to the nearest degree.

$$(a) \mathbf{u} = 4\mathbf{i} + 5\mathbf{j} \quad (b) \mathbf{v} = \begin{pmatrix} 0 \\ 10 \end{pmatrix}$$

Exercise 32

Express each of the following vectors in component form, giving each component correct to one decimal place.

- (a) The vector $\mathbf{p}$, with magnitude 12, that makes an angle of 100° with the positive x -direction.
- (b) The vector $\mathbf{q}$, with magnitude 27, that makes an angle of 200° with the positive x -direction.

Exercise 33

Express each of the following vectors in component form, assuming that $\mathbf{i}$ points east and $\mathbf{j}$ points north. Give each component correct to one decimal place.

- (a) The displacement $\mathbf{p}$, with magnitude 29 m and bearing 76° .
- (b) The velocity $\mathbf{v}$, with magnitude 55 ms^{-1} and bearing 115° .

Exercise 34

The displacement from Glasgow to Edinburgh is 69 km with a bearing of 81° . The displacement from Edinburgh to Stirling is 50 km with a bearing of 292° . Use components to find the magnitude (to the nearest km) and direction (as a bearing to the nearest degree) of the displacement from Glasgow to Stirling.

(Note that in this exercise you are being asked to use a different method to solve the problem in Exercise 24.)

Exercise 35

A bird has a speed in still air of 15 ms^{-1} . It is pointed in the direction of the bearing 140° but it flies in a wind of speed 8 ms^{-1} blowing from the south. Use components to find the resultant velocity of the bird in terms of its speed (in ms^{-1} to one decimal place) and bearing (to the nearest degree).

(Note that in this exercise you are being asked to use a different method to solve the problem in Exercise 25.)

7 Scalar product of two vectors

Exercise 36

For each of the following pairs of vectors, $\mathbf{u}$ and $\mathbf{v}$, find $\mathbf{u} \cdot \mathbf{v}$ and the angle between the vectors to the nearest degree.

(a) $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

(b) $\mathbf{u} = \begin{pmatrix} 7 \\ -1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$

Exercise 37

Two model boats are moving across a pond. Their velocities (in m s^{-1}) are represented by the vectors $\mathbf{a} = -2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{b} = 5\mathbf{i} - \mathbf{j}$, respectively. Find the angle between the paths of the two boats.

Solutions to exercises

Solution to Exercise 1

- (a) The distance between the points $(-1, 1)$ and $(4, 3)$ is

$$\begin{aligned} & \sqrt{(4 - (-1))^2 + (3 - 1)^2} \\ &= \sqrt{5^2 + 2^2} = \sqrt{29}. \end{aligned}$$

- (b) The distance between the points $(0, 6)$ and $(2, -2)$ is

$$\begin{aligned} & \sqrt{(2 - 0)^2 + (-2 - 6)^2} \\ &= \sqrt{2^2 + (-8)^2} = \sqrt{68} = 2\sqrt{17}. \end{aligned}$$

- (c) The distance between the points $(-7, -4)$ and $(-2, -1)$ is

$$\begin{aligned} & \sqrt{(-2 - (-7))^2 + (-1 - (-4))^2} \\ &= \sqrt{5^2 + 3^2} = \sqrt{34}. \end{aligned}$$

Solution to Exercise 2

The midpoint is

$$\left(\frac{-7 + (-2)}{2}, \frac{-4 + (-1)}{2} \right) = \left(-\frac{9}{2}, -\frac{5}{2} \right).$$

Solution to Exercise 3

The midpoint of the line segment joining the points is

$$\left(\frac{0 + 6}{2}, \frac{6 + (-2)}{2} \right) = (3, 2).$$

The gradient of the line segment is

$$\frac{(-2) - 6}{6 - 0} = \frac{-8}{6} = -\frac{4}{3}.$$

So the gradient of the perpendicular bisector is $\frac{3}{4}$.

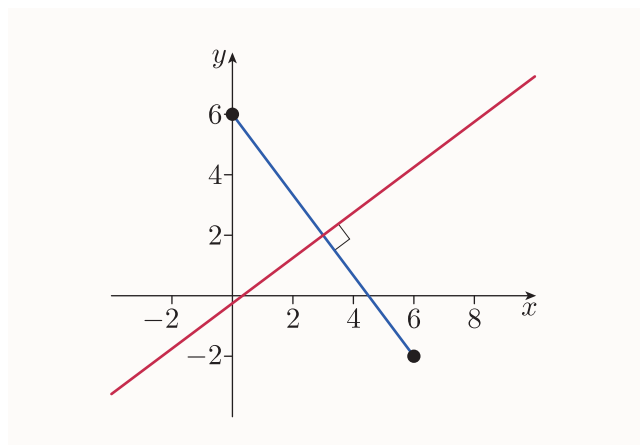
Hence (from the equation $y - y_1 = m(x - x_1)$) the equation of the perpendicular bisector is

$$y - 2 = \frac{3}{4}(x - 3);$$

that is,

$$y = \frac{3}{4}x - \frac{1}{4}.$$

(The line segment and the perpendicular bisector are shown below.)



Solution to Exercise 4

- (a) The circle with centre $(3, 2)$ and radius 5 has equation $(x - 3)^2 + (y - 2)^2 = 25$.

- (b) The circle with centre $(-\frac{1}{2}, 0)$ and radius $\frac{2}{3}$ has equation

$$(x - (-\frac{1}{2}))^2 + y^2 = (\frac{2}{3})^2;$$

that is,

$$(x + \frac{1}{2})^2 + y^2 = \frac{4}{9}.$$

- (c) The circle with centre $(-1, -1)$ and radius $\sqrt{7}$ has equation

$$(x - (-1))^2 + (y - (-1))^2 = (\sqrt{7})^2;$$

that is,

$$(x + 1)^2 + (y + 1)^2 = 7.$$

Solution to Exercise 5

- (a) The circle with equation $(x - 6)^2 + (y + 3)^2 = 121$ has centre $(6, -3)$ and radius $\sqrt{121} = 11$.

- (b) The equation is $3x^2 + 3y^2 = 9$.

Dividing through by 3 to put the equation in standard form gives $x^2 + y^2 = 3$.

So the centre is $(0, 0)$ and the radius is $\sqrt{3}$.

- (c) The equation is

$$\frac{(x + k)^2}{2} + \frac{(y + k)^2}{2} = h.$$

Multiplying through by 2 to put the equation in standard form gives $(x + k)^2 + (y + k)^2 = 2h$.

So the centre is $(-k, -k)$ and the radius is $\sqrt{2h}$.

Solution to Exercise 6

- (a) The equation is

$$x^2 + y^2 + 10y = 0.$$

Completing the square in the subexpression involving y^2 and y gives

$$x^2 + (y + 5)^2 - 25 = 0;$$

that is,

$$x^2 + (y + 5)^2 = 25.$$

This equation is of the form $(x - a)^2 + (y - b)^2 = r^2$ with $a = 0$, $b = -5$ and $r = 5$. So it is the equation of a circle with centre $(0, -5)$ and radius 5.

- (b) The equation is

$$x^2 + 6x + y^2 - 5y = -13.$$

Completing the square in the subexpression involving x^2 and x and in the subexpression involving y^2 and y gives

$$(x + 3)^2 - 9 + (y - \frac{5}{2})^2 - \frac{25}{4} = -13.$$

Simplifying gives

$$(x + 3)^2 + (y - \frac{5}{2})^2 = \frac{9}{4}.$$

This equation is of the form $(x - a)^2 + (y - b)^2 = r^2$ with $a = -3$, $b = \frac{5}{2}$ and $r = \sqrt{\frac{9}{4}} = \frac{3}{2}$. So it is the equation of a circle with centre $(-3, \frac{5}{2})$ and radius $\frac{3}{2}$.

- (c) The equation is

$$2x^2 - 4x + 2y^2 + 4y - 1 = 0.$$

Dividing through by 2 gives

$$x^2 - 2x + y^2 + 2y - \frac{1}{2} = 0.$$

Completing the square in the subexpression involving x^2 and x and in the subexpression involving y^2 and y gives

$$(x - 1)^2 - 1 + (y + 1)^2 - 1 - \frac{1}{2} = 0.$$

Simplifying gives

$$(x - 1)^2 + (y + 1)^2 = \frac{5}{2}.$$

This equation is of the form $(x - a)^2 + (y - b)^2 = r^2$ with $a = 1$, $b = -1$ and $r = \sqrt{\frac{5}{2}}$. So it is the equation of a circle with centre $(1, -1)$ and radius $\sqrt{\frac{5}{2}}$.

Solution to Exercise 7

- (a) The equation $(x - 3)^2 + y^2 = -5$ is not the equation of a circle, because when it is written in standard form, the term on the right-hand side is -5 , which is not the square of the radius of a circle.

- (b) The equation $x^2 - y^2 = 25$ is not the equation of a circle, because the coefficient of x^2 , which is 1, is not the same as the coefficient of y^2 , which is -1 .

- (c) The equation $(x + 5)^2 + (2y - 6)^2 = 100$ is not the equation of a circle, because, when the brackets are expanded, the coefficient of x^2 is 1, and this is not the same as the coefficient of y^2 , which is 4.

- (d) The equation $x^2 + y^2 - 6x = -5$ can be written as

$$(x - 3)^2 - 9 + y^2 = -5;$$

that is,

$$(x - 3)^2 + y^2 = 4.$$

This is the equation of a circle with centre $(3, 0)$ and radius 2.

- (e) The equation is

$$10x^2 + 20x + 10y^2 - 20y = 0.$$

Dividing through by 10 gives

$$x^2 + 2x + y^2 - 2y = 0.$$

Completing the squares gives

$$(x + 1)^2 - 1 + (y - 1)^2 - 1 = 0;$$

that is,

$$(x + 1)^2 + (y - 1)^2 = 2.$$

This is the equation of a circle with centre $(-1, 1)$ and radius $\sqrt{2}$.

- (f) The equation is

$$x^2 + 14x + y^2 + 14y + 98 = 0.$$

Completing the squares gives

$$(x + 7)^2 - 49 + (y + 7)^2 - 49 + 98 = 0;$$

that is,

$$(x + 7)^2 + (y + 7)^2 = 0.$$

This is not the equation of a circle because the radius would be 0.

Solution to Exercise 8

- (a) Substituting $x = 2$ into the equation of the circle gives

$$2^2 + 4 \times 2 + y^2 - 8y = 5$$

$$y^2 - 8y + 7 = 0$$

$$(y - 7)(y - 1) = 0.$$

So $y = 7$ or $y = 1$.

The points $(2, 7)$ and $(2, 1)$ lie on the circle.

- (b) Substituting $x = 3$ into the equation of the circle gives

$$3^2 + 4 \times 3 + y^2 - 8y = 5$$

$$y^2 - 8y + 16 = 0$$

$$(y - 4)(y - 4) = 0.$$

So $y = 4$.

There is only one point with x -coordinate 3 on the circle, namely $(3, 4)$.

- (c) Substituting $y = -2$ into the equation of the circle gives

$$x^2 + 4x + (-2)^2 - 8 \times (-2) = 5$$

$$x^2 + 4x + 15 = 0.$$

The discriminant of the quadratic on the left-hand side of this equation is

$$b^2 - 4ac = 16 - 4 \times 15 = -44.$$

Since it is negative, the quadratic equation has no real solutions.

Hence there are no points on the circle with y -coordinate -2 .

- (d) Substituting $y = 0$ into the equation of the circle gives

$$x^2 + 4x = 5$$

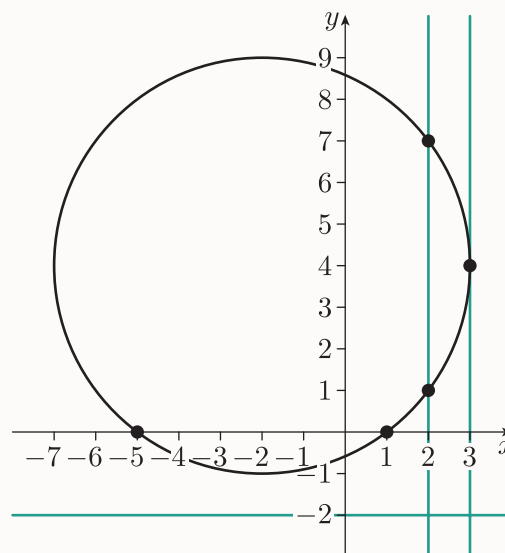
$$x^2 + 4x - 5 = 0$$

$$(x + 5)(x - 1) = 0.$$

So $x = -5$ or $x = 1$.

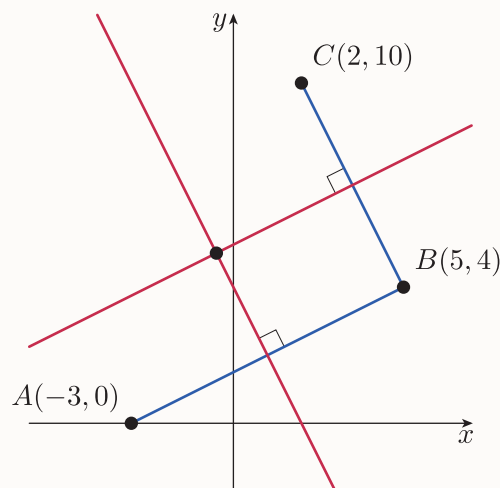
The points $(-5, 0)$ and $(1, 0)$ lie on the circle.

(The circle is shown below.)



Solution to Exercise 9

- (a) The points are shown on the diagram below.



First we find the perpendicular bisector of AB . The midpoint of AB is

$$\left(\frac{-3 + 5}{2}, \frac{0 + 4}{2} \right) = (1, 2).$$

The gradient of AB is

$$\frac{4 - 0}{5 - (-3)} = \frac{1}{2}.$$

So the gradient of the perpendicular bisector of AB is -2 .

Hence the equation of the perpendicular bisector of AB is

$$y - 2 = -2(x - 1)$$

$$y = -2x + 4.$$

Now we find the perpendicular bisector of BC . The midpoint of BC is

$$\left(\frac{2+5}{2}, \frac{10+4}{2}\right) = \left(\frac{7}{2}, 7\right).$$

The gradient of BC is

$$\frac{10-4}{2-5} = -2.$$

So the gradient of the perpendicular bisector of BC is $\frac{1}{2}$.

Hence the equation of the perpendicular bisector of BC is

$$\begin{aligned} y - 7 &= \frac{1}{2}\left(x - \frac{7}{2}\right) \\ y &= \frac{1}{2}x + \frac{21}{4}. \end{aligned}$$

To find the point of intersection of the perpendicular bisectors, which is the centre of the circle, we solve the equations $y = -2x + 4$ and $y = \frac{1}{2}x + \frac{21}{4}$ simultaneously.

Equating the expressions for y gives

$$\begin{aligned} -2x + 4 &= \frac{1}{2}x + \frac{21}{4} \\ -8x + 16 &= 2x + 21 \\ 10x &= -5 \\ x &= -\frac{1}{2}. \end{aligned}$$

Substituting into the equation

$y = -2x + 4$ gives

$$y = -2 \times \left(-\frac{1}{2}\right) + 4 = 5.$$

So the centre of the circle is $\left(-\frac{1}{2}, 5\right)$.

The radius, r , is the distance between the centre $\left(-\frac{1}{2}, 5\right)$ and any one of the given points, say $B(5, 4)$. So it is given by

$$\begin{aligned} r^2 &= \left(5 - \left(-\frac{1}{2}\right)\right)^2 + (4 - 5)^2 \\ &= \left(\frac{11}{2}\right)^2 + 1 = \frac{125}{4}. \end{aligned}$$

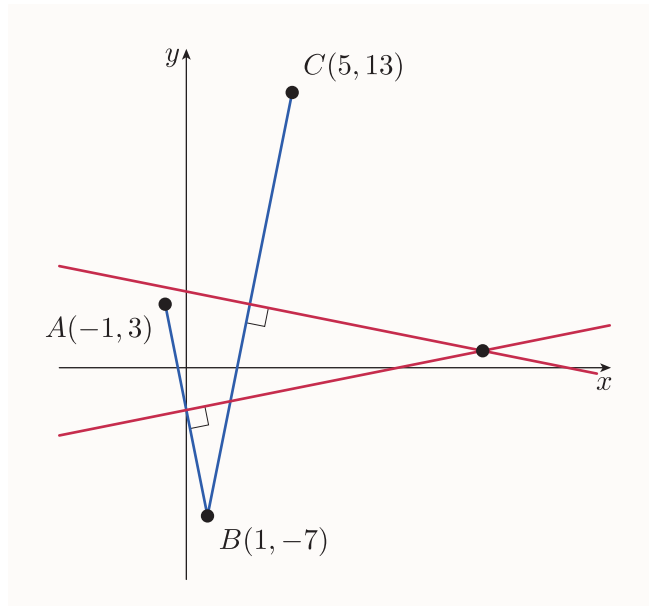
So the radius is

$$r = \sqrt{\frac{125}{4}} = \frac{5}{2}\sqrt{5}.$$

Hence the equation of the circle is

$$\left(x + \frac{1}{2}\right)^2 + (y - 5)^2 = \frac{125}{4}.$$

- (b) The points are shown on the diagram below.



First we find the perpendicular bisector of AB . The midpoint of AB is

$$\left(\frac{-1+1}{2}, \frac{3+(-7)}{2}\right) = (0, -2).$$

The gradient of AB is

$$\frac{-7-3}{1-(-1)} = -5.$$

So the gradient of the perpendicular bisector of AB is $\frac{1}{5}$.

Hence the equation of the perpendicular bisector of AB is

$$\begin{aligned} y - (-2) &= \frac{1}{5}(x - 0) \\ y &= \frac{1}{5}x - 2. \end{aligned}$$

Now we find the perpendicular bisector of BC . The midpoint of BC is

$$\left(\frac{1+5}{2}, \frac{-7+13}{2}\right) = (3, 3).$$

The gradient of BC is

$$\frac{13 - (-7)}{5 - 1} = 5.$$

So the gradient of the perpendicular bisector of BC is $-\frac{1}{5}$.

Hence the equation of the perpendicular bisector of BC is

$$\begin{aligned} y - 3 &= -\frac{1}{5}(x - 3) \\ y &= -\frac{1}{5}x + \frac{18}{5}. \end{aligned}$$

To find the point of intersection of the perpendicular bisectors, which is the centre of the circle, we solve the equations $y = \frac{1}{5}x - 2$ and $y = -\frac{1}{5}x + \frac{18}{5}$ simultaneously.

Equating the expressions for y gives

$$\frac{1}{5}x - 2 = -\frac{1}{5}x + \frac{18}{5}$$

$$x - 10 = -x + 18$$

$$2x = 28$$

$$x = 14.$$

Substituting into the equation $y = \frac{1}{5}x - 2$ gives

$$y = \frac{1}{5} \times 14 - 2 = \frac{4}{5}.$$

So the centre of the circle is $(14, \frac{4}{5})$.

The radius, r , is the distance between the centre $(14, \frac{4}{5})$ and any one of the given points, say $A(-1, 3)$. So it is given by

$$\begin{aligned} r^2 &= (14 - (-1))^2 + (\frac{4}{5} - 3)^2 \\ &= 15^2 + (-\frac{11}{5})^2 = \frac{5746}{25}. \end{aligned}$$

So the radius is

$$\sqrt{\frac{5746}{25}} = \frac{13}{5}\sqrt{34}.$$

Hence the equation of the circle is

$$(x - 14)^2 + (y - \frac{4}{5})^2 = \frac{5746}{25}.$$

Solution to Exercise 10

- (a) Substituting $x = 2$ into the equation of the circle gives

$$3^2 + (y - 4)^2 = 9$$

$$(y - 4)^2 = 0$$

$$y = 4.$$

So the line $x = 2$ intersects the circle at one point, $(2, 4)$.

- (b) Substituting $y = x$ into the equation of the circle gives

$$(x + 1)^2 + (x - 4)^2 = 9$$

$$x^2 + 2x + 1 + x^2 - 8x + 16 = 9$$

$$2x^2 - 6x + 8 = 0$$

$$x^2 - 3x + 4 = 0.$$

The discriminant of the quadratic on the left-hand side is

$$b^2 - 4ac = (-3)^2 - 4 \times 1 \times 4 = -7.$$

Since it is negative, the quadratic equation has no real solutions.

So there are no points that lie on both the line and the circle.

- (c) Substituting $y = x + 2$ into the equation of the circle gives

$$(x + 1)^2 + (x - 2)^2 = 9$$

$$x^2 + 2x + 1 + x^2 - 4x + 4 = 9$$

$$2x^2 - 2x - 4 = 0$$

$$x^2 - x - 2 = 0$$

$$(x + 1)(x - 2) = 0$$

$$x = -1 \text{ or } x = 2.$$

Substituting $x = -1$ into the equation of the line gives $y = -1 + 2 = 1$.

Substituting $x = 2$ into the equation of the line gives $y = 2 + 2 = 4$.

So the line intersects the circle at the points $(-1, 1)$ and $(2, 4)$.

Solution to Exercise 11

Solving the equations $y = x - 1$ and $y = x^2 + 2x - 3$ simultaneously gives

$$x - 1 = x^2 + 2x - 3$$

$$x^2 + x - 2 = 0$$

$$(x - 1)(x + 2) = 0$$

$$x = 1 \text{ or } x = -2.$$

Substituting $x = 1$ into the equation of the line gives $y = 1 - 1 = 0$.

Substituting $x = -2$ into the equation of the line gives $y = -2 - 1 = -3$.

So the line intersects the parabola at $(1, 0)$ and $(-2, -3)$.

Solution to Exercise 12

Solving the equations $y = 2x^2 + x - 4$ and $y = 2x^2 - x$ simultaneously gives

$$2x^2 + x - 4 = 2x^2 - x$$

$$2x = 4$$

$$x = 2.$$

Substituting $x = 2$ into the equation $y = 2x^2 - x$ gives $y = 2 \times 2^2 - 2 = 6$.

So the two parabolas intersect at one point, $(2, 6)$.

Solution to Exercise 13

Multiplying out the brackets in the equations gives

$$x^2 + 4x + 4 + y^2 + 2y + 1 = 26$$

$$x^2 - 6x + 9 + y^2 - 8y + 16 = 16.$$

Rearranging them gives

$$x^2 + y^2 + 4x + 2y - 21 = 0 \quad (1)$$

$$x^2 + y^2 - 6x - 8y + 9 = 0. \quad (2)$$

Subtracting equation (2) from equation (1) gives

$$10x + 10y - 30 = 0$$

$$x + y - 3 = 0$$

$$y = 3 - x.$$

Substituting $y = 3 - x$ into equation (1) gives

$$x^2 + 4x + (3 - x)^2 + 2(3 - x) - 21 = 0.$$

Simplifying and factorising gives

$$x^2 + 4x + 9 - 6x + x^2 + 6 - 2x - 21 = 0$$

$$2x^2 - 4x - 6 = 0$$

$$x^2 - 2x - 3 = 0$$

$$(x - 3)(x + 1) = 0$$

$$x = 3 \text{ or } x = -1.$$

Substituting $x = 3$ into the equation $y = 3 - x$ gives $y = 0$, and substituting $x = -1$ into the same equation gives $y = 4$.

So the two circles intersect at $(3, 0)$ and $(-1, 4)$.

Solution to Exercise 14

- (a) The distance between the points $(0, 7, 2)$ and $(3, 3, 14)$ is

$$\begin{aligned} & \sqrt{(3-0)^2 + (3-7)^2 + (14-2)^2} \\ &= \sqrt{3^2 + (-4)^2 + 12^2} \\ &= \sqrt{9 + 16 + 144} = \sqrt{169} = 13. \end{aligned}$$

- (b) The distance between the points $(8, -4, -2)$ and $(4, -1, 3)$ is

$$\begin{aligned} & \sqrt{(4-8)^2 + (-1-(-4))^2 + (3-(-2))^2} \\ &= \sqrt{(-4)^2 + 3^2 + 5^2} \\ &= \sqrt{16 + 9 + 25} = \sqrt{50} = 5\sqrt{2}. \end{aligned}$$

Solution to Exercise 15

The centre is $(0, 10, -4)$ and the radius is 9, so the equation is

$$(x - 0)^2 + (y - 10)^2 + (z - (-4))^2 = 9^2;$$

that is,

$$x^2 + (y - 10)^2 + (z + 4)^2 = 81.$$

Solution to Exercise 16

The equation of the sphere is

$$(x + 4)^2 + (y - 7)^2 + z^2 = 121,$$

so the centre is $(-4, 7, 0)$ and the radius is $\sqrt{121} = 11$.

Solution to Exercise 17

- (a) Completing the squares in the equation

$$x^2 + y^2 + z^2 - 2x + 14y = 50$$

gives

$$(x - 1)^2 - 1 + (y + 7)^2 - 49 + z^2 = 50;$$

that is,

$$(x - 1)^2 + (y + 7)^2 + z^2 = 100.$$

This is the equation of a sphere with centre $(1, -7, 0)$ and radius 10.

- (b) The equation

$$2x^2 - y^2 + 3z^2 - 2x + 8y + 6z = 73$$

is not the equation of a sphere, because the coefficients of x^2 , y^2 and z^2 are not the same.

- (c) Completing the squares in the equation

$$x^2 + y^2 + z^2 - 4x + 6y + 10z + 100 = 0$$

gives

$$\begin{aligned} & (x - 2)^2 - 4 + (y + 3)^2 - 9 \\ & + (z + 5)^2 - 25 + 100 = 0; \end{aligned}$$

that is,

$$(x - 2)^2 + (y + 3)^2 + (z + 5)^2 = -62.$$

This is not the equation of a sphere, because the constant term on the right-hand side is negative and so is not the square of the radius of a sphere.

- (d) Dividing the equation

$$5x^2 + 5y^2 + 5z^2 + 10x - 20y + 30z = 0$$

through by 5 gives

$$x^2 + y^2 + z^2 + 2x - 4y + 6z = 0.$$

Completing the squares gives

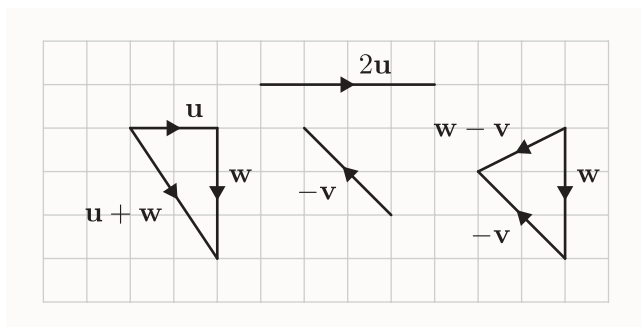
$$(x+1)^2 - 1 + (y-2)^2 - 4 + (z+3)^2 - 9 = 0;$$

that is,

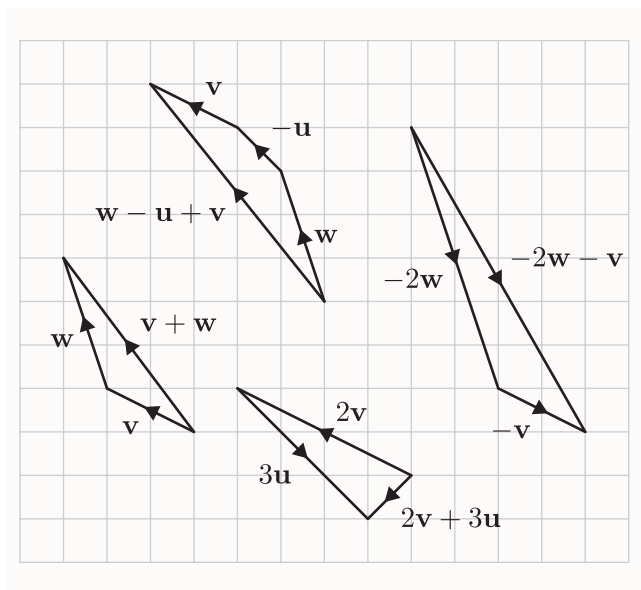
$$(x+1)^2 + (y-2)^2 + (z+3)^2 = 14.$$

This is the equation of a sphere with centre $(-1, 2, -3)$ and radius $\sqrt{14}$.

Solution to Exercise 18



Solution to Exercise 19



Solution to Exercise 20

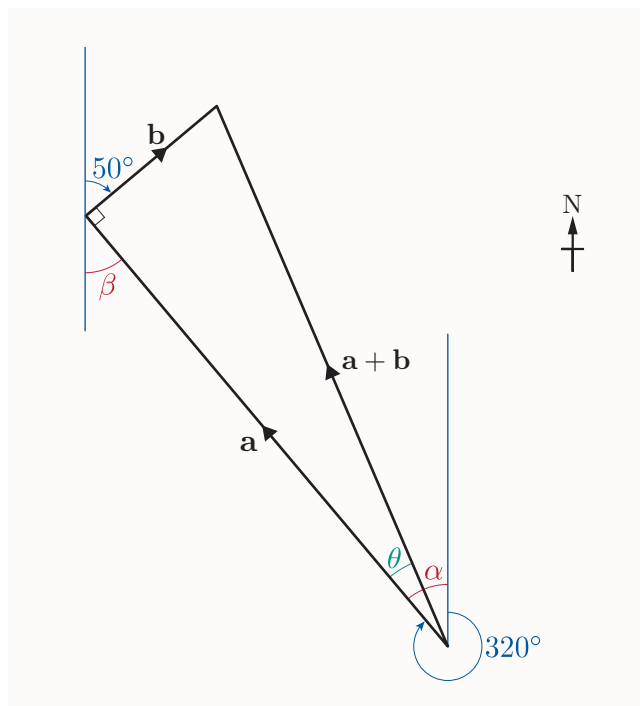
$$\begin{aligned} 7(\mathbf{a} + \mathbf{b}) - (\mathbf{a} - \mathbf{c}) + 6(\mathbf{b} - \mathbf{a} + 2\mathbf{c}) \\ = 7\mathbf{a} + 7\mathbf{b} - \mathbf{a} + \mathbf{c} + 6\mathbf{b} - 6\mathbf{a} + 12\mathbf{c} \\ = 13\mathbf{b} + 13\mathbf{c} \end{aligned}$$

Solution to Exercise 21

$$\begin{aligned} 3(\mathbf{a} + 2\mathbf{x}) &= 4(\mathbf{b} - \mathbf{a}) + \mathbf{a} \\ 3\mathbf{a} + 6\mathbf{x} &= 4\mathbf{b} - 4\mathbf{a} + \mathbf{a} \\ 6\mathbf{x} &= 4\mathbf{b} - 6\mathbf{a} \\ \mathbf{x} &= \frac{2}{3}\mathbf{b} - \mathbf{a} \end{aligned}$$

Solution to Exercise 22

Represent the first part of the orienteer's run by the vector $\mathbf{a}$, and the second part by $\mathbf{b}$. The resultant displacement is $\mathbf{a} + \mathbf{b}$, as shown in the diagram.



The angle marked α is given by

$$\alpha = 360^\circ - 320^\circ = 40^\circ.$$

Also $\alpha = \beta$, because these angles are alternate angles for the two parallel north-south lines. So $\beta = 40^\circ$ and hence the angle between $\mathbf{a}$ and $\mathbf{b}$ is

$$180^\circ - 50^\circ - 40^\circ = 90^\circ,$$

as marked.

By Pythagoras' theorem,

$$|\mathbf{a} + \mathbf{b}| = \sqrt{|\mathbf{a}|^2 + |\mathbf{b}|^2} = \sqrt{240^2 + 70^2} = 250.$$

The angle marked θ is given by

$$\tan \theta = \frac{|\mathbf{b}|}{|\mathbf{a}|} = \frac{70}{240},$$

which gives

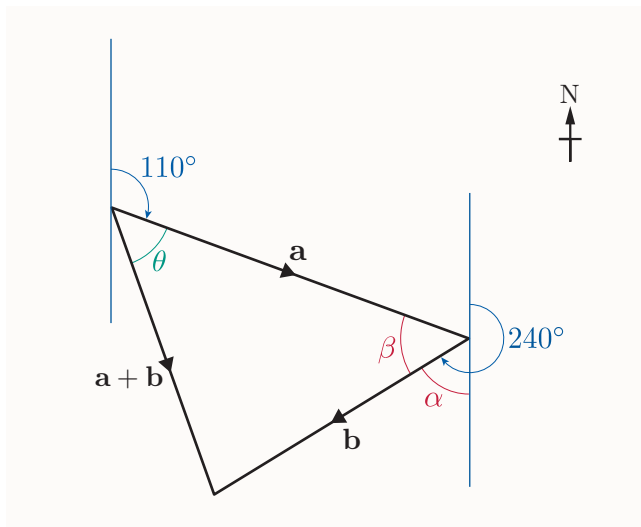
$$\begin{aligned} \theta &= \tan^{-1} \left(\frac{70}{240} \right) = 16.26 \dots^\circ \\ &= 16^\circ \text{ (to the nearest degree).} \end{aligned}$$

So the bearing of $\mathbf{a} + \mathbf{b}$ is $320^\circ + 16^\circ = 336^\circ$ (to the nearest degree).

The orienteer's resultant displacement has magnitude 250 m and a bearing of 336° .

Solution to Exercise 23

Represent the first part of the motion by the vector $\mathbf{a}$ and the second part by the vector $\mathbf{b}$. The resultant displacement is $\mathbf{a} + \mathbf{b}$, as shown in the diagram.



The angle marked α is given by

$$\alpha = 240^\circ - 180^\circ = 60^\circ.$$

The total angle $\alpha + \beta$ and the angle marked as 110° are alternate angles for the two north-south parallel lines. So $\alpha + \beta = 110^\circ$ and hence

$$\beta = 110^\circ - 60^\circ = 50^\circ.$$

By the cosine rule,

$$\begin{aligned} |\mathbf{a} + \mathbf{b}|^2 &= |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2 \times |\mathbf{a}| \times |\mathbf{b}| \times \cos 50^\circ \\ &= 15^2 + 12^2 - 2 \times 15 \times 12 \times \cos 50^\circ \\ &= 137.59 \dots, \end{aligned}$$

so

$$\begin{aligned} |\mathbf{a} + \mathbf{b}| &= \sqrt{137.59 \dots} \\ &= 11.73 \dots = 11.7 \text{ (to 1 d.p.)}. \end{aligned}$$

To find the bearing of $\mathbf{a} + \mathbf{b}$, we first find the angle θ . By the sine rule,

$$\frac{\sin \theta}{|\mathbf{b}|} = \frac{\sin 50^\circ}{|\mathbf{a} + \mathbf{b}|},$$

so

$$\sin \theta = \frac{12 \sin 50^\circ}{11.73 \dots} = 0.783 \dots,$$

which gives

$$\theta = 51.59 \dots^\circ$$

or

$$\theta = 180^\circ - 51.59 \dots^\circ = 128.40 \dots^\circ.$$

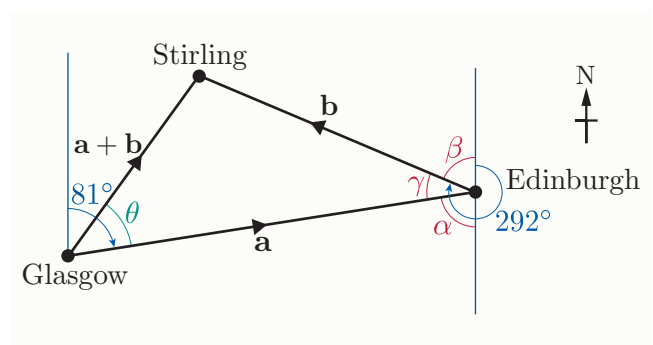
Since θ is opposite the side of the triangle formed by $\mathbf{b}$, and this is not the longest side, θ is not the largest angle. Hence $\theta = 52^\circ$, to the nearest degree.

So the bearing of $\mathbf{a} + \mathbf{b}$ is $110^\circ + 52^\circ = 162^\circ$, to the nearest degree.

The resultant displacement of the ball has magnitude 11.7 cm and bearing 162° .

Solution to Exercise 24

Let the displacement from Glasgow to Edinburgh be $\mathbf{a}$ and the displacement from Edinburgh to Stirling be $\mathbf{b}$. Then the resultant displacement from Glasgow to Stirling is $\mathbf{a} + \mathbf{b}$, as shown in the diagram.



The angle marked α is 81° , because it and the bearing of Edinburgh from Glasgow are alternate angles for the north-south parallel lines. Also

$$\beta = 360^\circ - 292^\circ = 68^\circ.$$

Hence

$$\gamma = 180^\circ - 68^\circ - 81^\circ = 31^\circ.$$

The distance from Glasgow to Edinburgh is 69 km and the distance from Edinburgh to Stirling is 50 km.

By the cosine rule,

$$\begin{aligned} |\mathbf{a} + \mathbf{b}|^2 &= |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2 \times |\mathbf{a}| \times |\mathbf{b}| \times \cos 31^\circ \\ &= 69^2 + 50^2 - 2 \times 69 \times 50 \times \cos 31^\circ \\ &= 1346.54 \dots, \end{aligned}$$

so

$$\begin{aligned} |\mathbf{a} + \mathbf{b}| &= \sqrt{1346.54 \dots} \\ &= 36.69 \dots = 37 \text{ (to the nearest km)}. \end{aligned}$$

We now find the angle θ . By the sine rule,

$$\frac{\sin \theta}{|\mathbf{b}|} = \frac{\sin 31^\circ}{|\mathbf{a} + \mathbf{b}|},$$

so

$$\sin \theta = \frac{50 \sin 31^\circ}{36.39 \dots} = 0.701 \dots,$$

which gives

$$\theta = 44.56 \dots^\circ$$

or

$$\theta = 180^\circ - 44.56 \dots^\circ = 135.43 \dots^\circ.$$

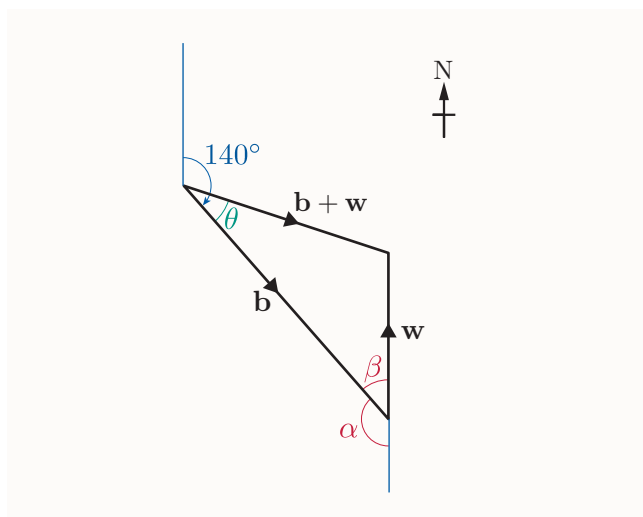
Since θ is opposite the side of the triangle formed by $\mathbf{b}$, and this is not the longest side, θ is not the largest angle. Hence $\theta = 45^\circ$, to the nearest degree.

So the bearing of Stirling from Glasgow is $81^\circ - 45^\circ = 36^\circ$, to the nearest degree.

The displacement of Stirling from Glasgow is 37 km on a bearing of 36° .

Solution to Exercise 25

Let $\mathbf{b}$ be the velocity of the bird in still air and let $\mathbf{w}$ be the velocity of the wind. The resultant velocity of the bird is $\mathbf{b} + \mathbf{w}$, as shown in the diagram.



The angle marked α is 140° , because it and the bearing of $\mathbf{b} + \mathbf{w}$ are alternate angles for the north-south parallel lines. So

$$\beta = 180^\circ - 140^\circ = 40^\circ.$$

By the cosine rule,

$$\begin{aligned} |\mathbf{b} + \mathbf{w}|^2 &= |\mathbf{b}|^2 + |\mathbf{w}|^2 - 2 \times |\mathbf{b}| \times |\mathbf{w}| \times \cos 40^\circ \\ &= 15^2 + 8^2 - 2 \times 15 \times 8 \times \cos 40^\circ \\ &= 105.14 \dots, \end{aligned}$$

so

$$\begin{aligned} |\mathbf{b} + \mathbf{w}| &= \sqrt{105.14 \dots} \\ &= 10.25 \dots = 10.3 \text{ (to 1 d.p.)}. \end{aligned}$$

We now find the angle θ . By the sine rule,

$$\frac{\sin \theta}{|\mathbf{w}|} = \frac{\sin 40^\circ}{|\mathbf{b} + \mathbf{w}|},$$

so

$$\sin \theta = \frac{8 \sin 40^\circ}{10.25 \dots} = 0.501 \dots,$$

which gives

$$\theta = 30.09 \dots^\circ$$

or

$$\theta = 180^\circ - 30.09 \dots^\circ = 149.90 \dots^\circ.$$

Since θ is opposite the side of the triangle formed by $\mathbf{w}$, and this is not the longest side, θ is not the largest angle. Hence $\theta = 30^\circ$, to the nearest degree.

So the bearing of $\mathbf{b} + \mathbf{w}$ is $140^\circ - 30^\circ = 110^\circ$, to the nearest degree.

The bird's resultant speed is 10.3 m s^{-1} on a bearing of 110° .

Solution to Exercise 26

$$\begin{aligned} \mathbf{u} &= -\mathbf{i} + \mathbf{j}, \\ \mathbf{v} &= -\mathbf{i} - 3\mathbf{j}, \\ \mathbf{w} &= -4\mathbf{i}. \end{aligned}$$

Solution to Exercise 27

- $\mathbf{a} + \mathbf{b} = (\mathbf{i} + 2\mathbf{j}) + (-3\mathbf{i} + \mathbf{j}) = -2\mathbf{i} + 3\mathbf{j}$
- $\mathbf{a} - \mathbf{b} - \mathbf{c} = (\mathbf{i} + 2\mathbf{j}) - (-3\mathbf{i} + \mathbf{j}) - (-2\mathbf{i} - 5\mathbf{j})$
 $= 6\mathbf{i} + 6\mathbf{j}$
- $2\mathbf{c} = 2(-2\mathbf{i} - 5\mathbf{j}) = -4\mathbf{i} - 10\mathbf{j}$
- $\frac{1}{3}\mathbf{b} = \frac{1}{3}(-3\mathbf{i} + \mathbf{j}) = -\mathbf{i} + \frac{1}{3}\mathbf{j}$
- $\mathbf{c} - 3\mathbf{b} = (-2\mathbf{i} - 5\mathbf{j}) - 3(-3\mathbf{i} + \mathbf{j})$
 $= -2\mathbf{i} - 5\mathbf{j} + 9\mathbf{i} - 3\mathbf{j} = 7\mathbf{i} - 8\mathbf{j}$
- $2\mathbf{a} + 5\mathbf{b} - 7\mathbf{c}$
 $= 2(\mathbf{i} + 2\mathbf{j}) + 5(-3\mathbf{i} + \mathbf{j}) - 7(-2\mathbf{i} - 5\mathbf{j})$
 $= 2\mathbf{i} + 4\mathbf{j} - 15\mathbf{i} + 5\mathbf{j} + 14\mathbf{i} + 35\mathbf{j}$
 $= \mathbf{i} + 44\mathbf{j}$

Solution to Exercise 28

- $-3 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$
- $4 \begin{pmatrix} -2 \\ 5 \end{pmatrix} - 2 \begin{pmatrix} 3 \\ -7 \end{pmatrix} = \begin{pmatrix} -14 \\ 34 \end{pmatrix}$
- $2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - 3 \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -10 \\ -3 \\ 1 \end{pmatrix}$

Solution to Exercise 29

We have $\vec{OA} = -2\mathbf{i} - \mathbf{j}$ and $\vec{OB} = 5\mathbf{i}$.

$$\begin{aligned} \text{(a) } \vec{AB} &= \vec{OB} - \vec{OA} \\ &= 5\mathbf{i} - (-2\mathbf{i} - \mathbf{j}) \\ &= 7\mathbf{i} + \mathbf{j} \end{aligned}$$

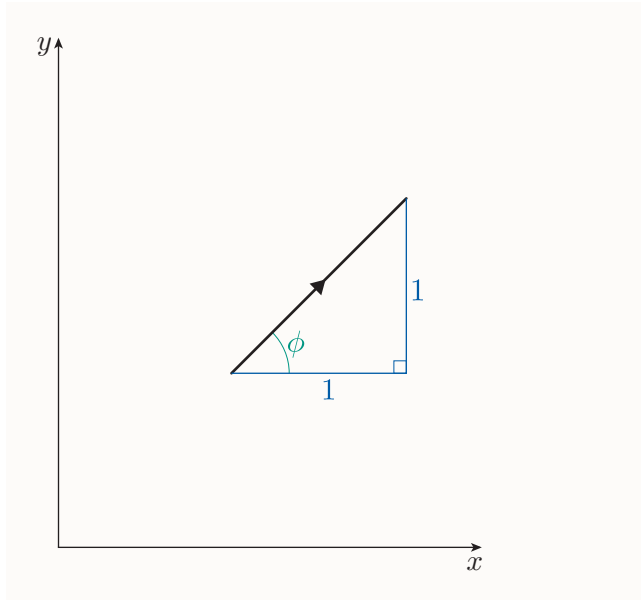
$$\begin{aligned} \text{(b) } \vec{BA} &= \vec{OA} - \vec{OB} \\ &= (-2\mathbf{i} - \mathbf{j}) - 5\mathbf{i} \\ &= -7\mathbf{i} - \mathbf{j} \end{aligned}$$

(Notice that, as expected, $\vec{BA} = -\vec{AB}$.)

Solution to Exercise 30

(a) The magnitude of the vector $\mathbf{a} = \mathbf{i} + \mathbf{j}$ is

$$|\mathbf{a}| = \sqrt{1^2 + 1^2} = \sqrt{2}.$$



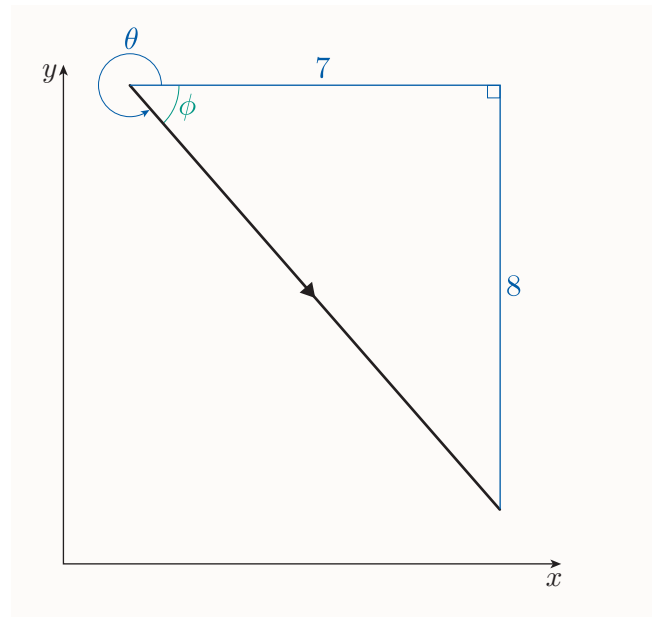
From the diagram,

$$\phi = \tan^{-1}\left(\frac{1}{1}\right) = 45^\circ.$$

So the angle that the vector $\mathbf{a}$ makes with the positive x -direction is 45° .

(b) The magnitude of the vector $\mathbf{b} = 7\mathbf{i} - 8\mathbf{j}$ is

$$|\mathbf{b}| = \sqrt{7^2 + (-8)^2} = \sqrt{113}.$$



From the diagram,

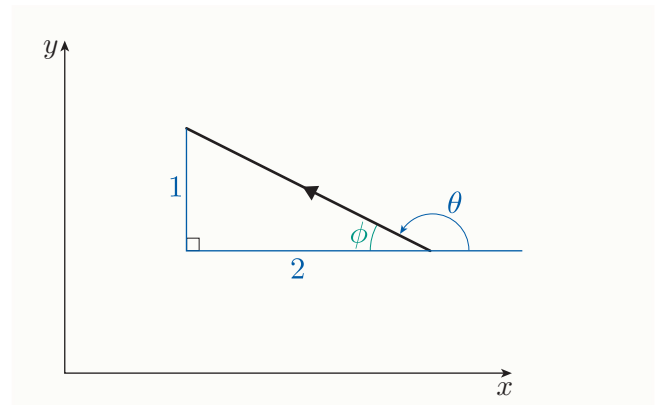
$$\phi = \tan^{-1}\left(\frac{8}{7}\right) = 48.81\dots^\circ.$$

So the angle that the vector $\mathbf{b}$ makes with the positive x -direction is

$$\begin{aligned} &360^\circ - 48.81\dots^\circ \\ &= 311.18\dots^\circ \\ &= 311^\circ \text{ (to the nearest degree).} \end{aligned}$$

(c) The magnitude of the vector $\mathbf{c} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ is

$$|\mathbf{c}| = \sqrt{(-2)^2 + 1^2} = \sqrt{5}.$$



From the diagram,

$$\phi = \tan^{-1}\left(\frac{1}{2}\right) = 26.56\dots^\circ.$$

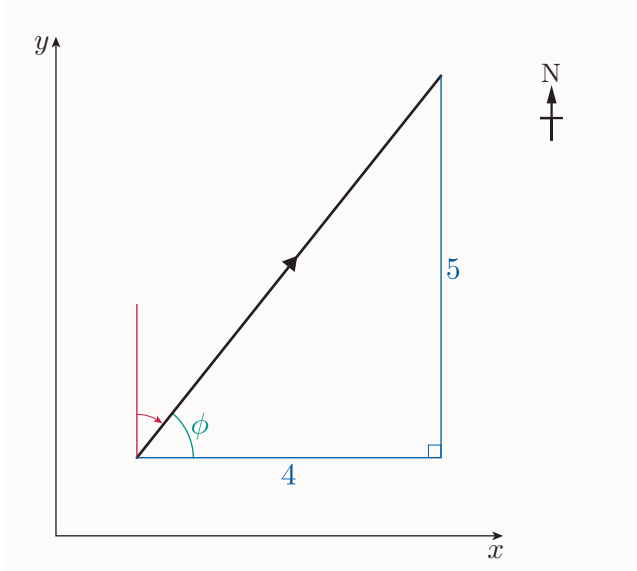
So the angle θ that the vector $\mathbf{b}$ makes with the positive x -direction is

$$\begin{aligned} &180^\circ - 26.56\dots^\circ \\ &= 153.43\dots^\circ \\ &= 153^\circ \text{ (to the nearest degree).} \end{aligned}$$

Solution to Exercise 31

- (a) The magnitude of the vector $\mathbf{u} = 4\mathbf{i} + 5\mathbf{j}$ is

$$\begin{aligned} |\mathbf{u}| &= \sqrt{4^2 + 5^2} = \sqrt{41} \\ &= 6.40\dots = 6.4 \text{ ms}^{-1} \text{ (to 1 d.p.)}. \end{aligned}$$



From the diagram,

$$\phi = \tan^{-1}\left(\frac{5}{4}\right) = 51.34\dots^\circ.$$

So the bearing of $\mathbf{u}$ is

$$\begin{aligned} 90^\circ - 51.34\dots^\circ &= 38.65\dots^\circ \\ &= 39^\circ \text{ (to the nearest degree)}. \end{aligned}$$

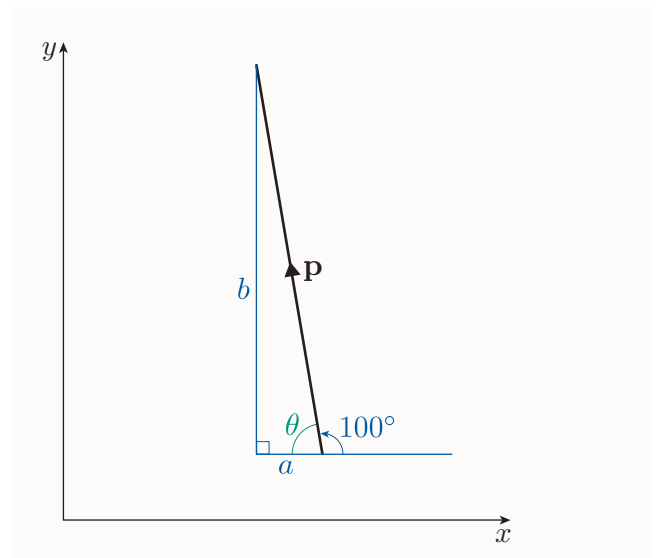
- (b) The magnitude of the vector $\mathbf{v} = \begin{pmatrix} 0 \\ 10 \end{pmatrix}$ is

$$|\mathbf{v}| = \sqrt{0^2 + 10^2} = 10 \text{ ms}^{-1}.$$

The $\mathbf{i}$ -component of $\mathbf{v}$ is zero and its $\mathbf{j}$ -component is positive, so $\mathbf{v}$ points due north. Hence its bearing is 0° .

Solution to Exercise 32

- (a)



The acute angle θ is $180^\circ - 100^\circ = 80^\circ$. So

$$\begin{aligned} a &= |\mathbf{p}| \cos \theta \\ &= 12 \cos 80^\circ \\ &= 2.08\dots = 2.1 \text{ (to 1 d.p.)} \end{aligned}$$

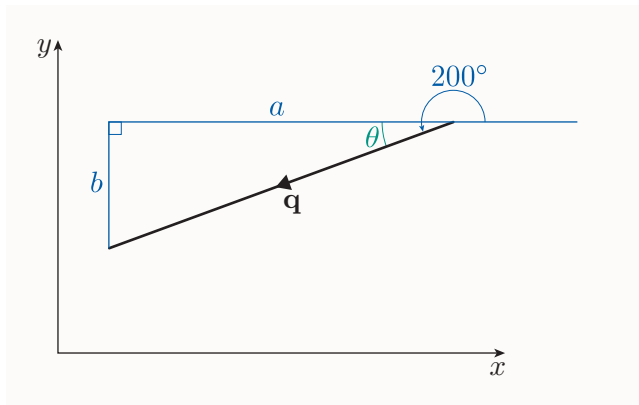
and

$$\begin{aligned} b &= |\mathbf{p}| \sin \theta \\ &= 12 \sin 80^\circ \\ &= 11.81\dots = 11.8 \text{ (to 1 d.p.)}. \end{aligned}$$

From the tail to the tip of $\mathbf{p}$ the x -coordinate decreases and the y -coordinate increases, so the $\mathbf{i}$ -component is negative and the $\mathbf{j}$ -component is positive. So

$$\mathbf{p} = -2.1\mathbf{i} + 11.8\mathbf{j} \text{ (to 1 d.p.)}.$$

(b)



The acute angle θ is $200^\circ - 180^\circ = 20^\circ$. So

$$\begin{aligned} a &= |\mathbf{q}| \cos \theta \\ &= 27 \cos 20^\circ \\ &= 25.37 \dots = 25.4 \text{ (to 1 d.p.)} \end{aligned}$$

and

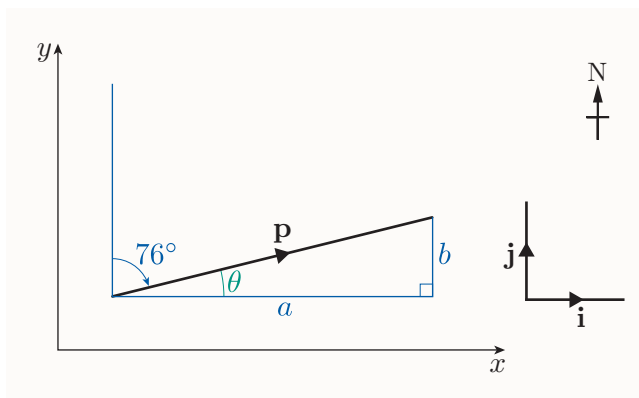
$$\begin{aligned} b &= |\mathbf{q}| \sin \theta \\ &= 27 \sin 20^\circ \\ &= 9.23 \dots = 9.2 \text{ (to 1 d.p.)}. \end{aligned}$$

From the tail to the tip of $\mathbf{q}$ the x -coordinate and the y -coordinate both decrease, so the $\mathbf{i}$ -component and the $\mathbf{j}$ -component are both negative. So

$$\mathbf{q} = -25.4\mathbf{i} - 9.2\mathbf{j} \text{ (to 1 d.p.)}.$$

Solution to Exercise 33

(a)



The angle θ is $90^\circ - 76^\circ = 14^\circ$. So

$$\begin{aligned} a &= |\mathbf{p}| \cos \theta \\ &= 29 \cos 14^\circ \\ &= 28.13 \dots = 28.1 \text{ (to 1 d.p.)} \end{aligned}$$

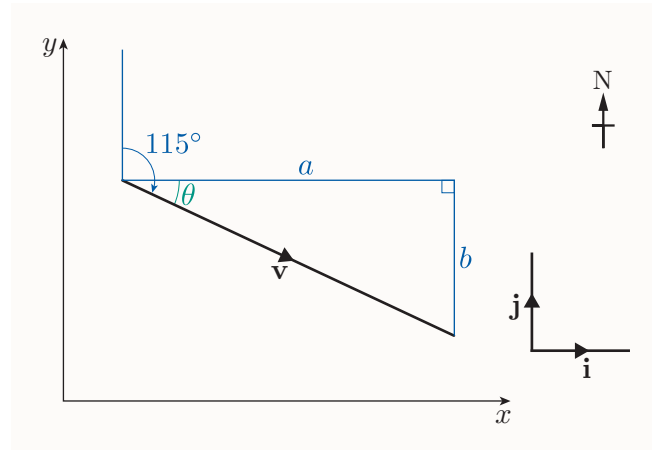
and

$$\begin{aligned} b &= |\mathbf{p}| \sin \theta \\ &= 29 \sin 14^\circ \\ &= 7.01 \dots = 7.0 \text{ (to 1 d.p.)}. \end{aligned}$$

From the tail to the tip of $\mathbf{p}$ the x -coordinate and the y -coordinate both increase, so the $\mathbf{i}$ -component and the $\mathbf{j}$ -component are both positive. So

$$\mathbf{p} = 28.1\mathbf{i} + 7.0\mathbf{j} \text{ (in m, to 1 d.p.)}.$$

(b)



The angle θ is $115^\circ - 90^\circ = 25^\circ$. So

$$\begin{aligned} a &= |\mathbf{v}| \cos \theta \\ &= 55 \cos 25^\circ \\ &= 49.84 \dots = 49.8 \text{ (to 1 d.p.)} \end{aligned}$$

and

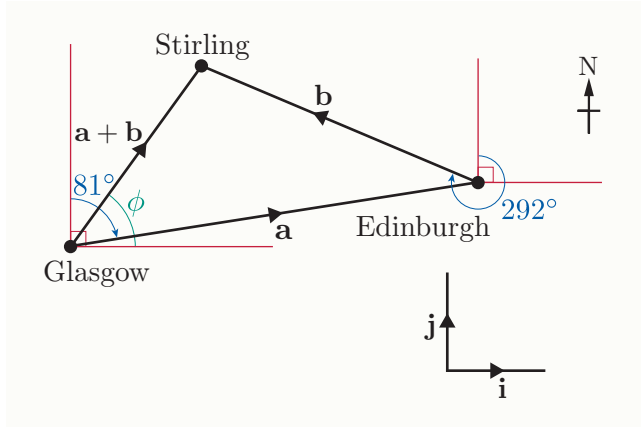
$$\begin{aligned} b &= |\mathbf{v}| \sin \theta \\ &= 55 \sin 25^\circ \\ &= 23.24 \dots = 23.2 \text{ (to 1 d.p.)}. \end{aligned}$$

From the tail to the tip of $\mathbf{v}$ the x -coordinate increases and the y -coordinate decreases, so the $\mathbf{i}$ -component is positive and the $\mathbf{j}$ -component is negative. So

$$\mathbf{v} = 49.8\mathbf{i} - 23.2\mathbf{j} \text{ (in m s}^{-1}\text{, to 1 d.p.)}.$$

Solution to Exercise 34

Let the displacement from Glasgow to Edinburgh be $\mathbf{a}$ and the displacement from Edinburgh to Stirling be $\mathbf{b}$. Then the resultant displacement from Glasgow to Stirling is $\mathbf{a} + \mathbf{b}$, as shown in the diagram.



Here $|\mathbf{a}| = 69$ and $\mathbf{a}$ makes an angle of $90^\circ - 81^\circ = 9^\circ$ with the positive x -direction, so

$$\mathbf{a} = 69 \cos 9^\circ \mathbf{i} + 69 \sin 9^\circ \mathbf{j}.$$

Also $|\mathbf{b}| = 50$ and $\mathbf{b}$ makes an angle of $90^\circ + (360^\circ - 292^\circ) = 158^\circ$ with the positive x -direction, so

$$\mathbf{b} = 50 \cos 158^\circ \mathbf{i} + 50 \sin 158^\circ \mathbf{j}.$$

Thus

$$\begin{aligned} \mathbf{a} + \mathbf{b} &= 69 \cos 9^\circ \mathbf{i} + 69 \sin 9^\circ \mathbf{j} \\ &\quad + 50 \cos 158^\circ \mathbf{i} + 50 \sin 158^\circ \mathbf{j} \\ &= 21.79 \dots \mathbf{i} + 29.52 \dots \mathbf{j}. \end{aligned}$$

Hence

$$\begin{aligned} |\mathbf{a} + \mathbf{b}| &= \sqrt{(21.79 \dots)^2 + (29.52 \dots)^2} \\ &= 36.69 \dots \end{aligned}$$

So the magnitude of the displacement from Glasgow to Stirling is 37 km (to the nearest km).

The diagram and the components of $\mathbf{a} + \mathbf{b}$ calculated above give

$$\tan \phi = \frac{29.52 \dots}{21.79 \dots}$$

so

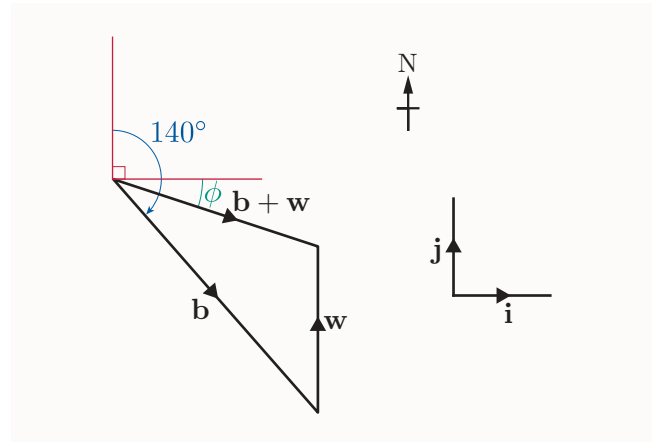
$$\phi = 53.56 \dots^\circ.$$

So the bearing of the displacement from Glasgow to Stirling is

$$\begin{aligned} 90^\circ - 53.56 \dots^\circ \\ = 36.43 \dots^\circ = 36^\circ \text{ (to the nearest degree).} \end{aligned}$$

Solution to Exercise 35

Let $\mathbf{b}$ be the velocity of the bird in still air and let $\mathbf{w}$ be the velocity of the wind. Then the resultant velocity is $\mathbf{b} + \mathbf{w}$, as shown in the diagram.



Here $|\mathbf{b}| = 15$ and $\mathbf{b}$ makes an angle of $90^\circ - 140^\circ = -50^\circ$ with the positive x -direction, so

$$\mathbf{b} = 15 \cos(-50^\circ) \mathbf{i} + 15 \sin(-50^\circ) \mathbf{j}.$$

Also $|\mathbf{w}| = 8$ and $\mathbf{w}$ lies due north in the $\mathbf{j}$ direction, so

$$\mathbf{w} = 8\mathbf{j}.$$

Thus

$$\begin{aligned} \mathbf{b} + \mathbf{w} &= 15 \cos(-50^\circ) \mathbf{i} + 15 \sin(-50^\circ) \mathbf{j} + 8\mathbf{j} \\ &= 9.64 \dots \mathbf{i} - 3.49 \dots \mathbf{j}. \end{aligned}$$

Hence

$$\begin{aligned} |\mathbf{b} + \mathbf{w}| &= \sqrt{(9.64 \dots)^2 + (-3.49 \dots)^2} \\ &= 10.25 \dots \end{aligned}$$

So the bird's speed is 10.3 m s^{-1} (to 1 d.p.).

The diagram and the components of $\mathbf{b} + \mathbf{w}$ calculated above give

$$\tan \phi = \frac{3.49 \dots}{9.64 \dots}$$

so

$$\phi = 19.90 \dots^\circ.$$

So the bird's velocity has bearing

$$\begin{aligned} 90^\circ + 19.90 \dots^\circ \\ = 109.90 \dots^\circ = 110^\circ \text{ (to the nearest degree).} \end{aligned}$$

Solution to Exercise 36

(a) We have

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= (\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) \cdot (3\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \\ &= (1 \times 3) + (-2 \times 2) + (2 \times (-1)) \\ &= -3,\end{aligned}$$

$$|\mathbf{u}| = \sqrt{1^2 + (-2)^2 + 2^2} = 3,$$

$$|\mathbf{v}| = \sqrt{3^2 + 2^2 + (-1)^2} = \sqrt{14}.$$

So, if θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| |\mathbf{v}|} = \frac{-3}{3 \times \sqrt{14}} = -\frac{1}{\sqrt{14}}$$

and hence

$$\theta = \cos^{-1} \left(-\frac{1}{\sqrt{14}} \right) = 105.50 \dots^\circ.$$

So the angle between the vectors is 106° (to the nearest degree).

(b) We have

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= \begin{pmatrix} 7 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 3 \end{pmatrix} \\ &= (7 \times 3) + (-1 \times 3) = 18,\end{aligned}$$

$$|\mathbf{u}| = \sqrt{7^2 + (-1)^2} = \sqrt{50} = 5\sqrt{2},$$

$$|\mathbf{v}| = \sqrt{3^2 + 3^2} = \sqrt{18} = 3\sqrt{2}.$$

So, if θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| |\mathbf{v}|} = \frac{18}{5\sqrt{2} \times 3\sqrt{2}} = \frac{18}{30} = \frac{3}{5}$$

and hence

$$\theta = \cos^{-1} \left(\frac{3}{5} \right) = 53.13 \dots^\circ.$$

So the angle between the vectors is 53° (to the nearest degree).

Solution to Exercise 37

We have

$$\begin{aligned}\mathbf{a} \cdot \mathbf{b} &= (-2\mathbf{i} + 3\mathbf{j}) \cdot (5\mathbf{i} - \mathbf{j}) \\ &= ((-2) \times 5) + (3 \times (-1)) = -13,\end{aligned}$$

$$|\mathbf{a}| = \sqrt{(-2)^2 + 3^2} = \sqrt{13},$$

$$|\mathbf{b}| = \sqrt{5^2 + (-1)^2} = \sqrt{26}.$$

So, if θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, then

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|} = \frac{-13}{\sqrt{13} \times \sqrt{26}} = -\frac{1}{\sqrt{2}}$$

and hence

$$\theta = \cos^{-1} \left(-\frac{1}{\sqrt{2}} \right) = 135^\circ.$$

So the angle between the paths of the boats is 135° .